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BOUNDARY VALUE PROBLEMS FOR TURBULENCE MODEL EQUATIONS

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Boundary value problems for a system of equations serving as a mathematical model of the turbulent motion of a liquid or gas are investigated. The model in question was introduced by Burgers in [2]. Section 1 contains a proof of the existence of at least one smooth time-periodic solution of the first boundary value problem for this system. This is accomplished with the aid of the Leray-Schauder topological principle [2] concerning the existence of fixed points of completely continuous operators. The existence theorem is prefaced by a derivation of the prior estimates of the solution of the problem which are necessary for the realization of the topological principle. Section 2 deals with the first boundary value problem with initial conditions and with the Cauchy problems for the turbulence model equations.

Let us begin by introducing some symbols. We denote the interval $(0, 1)$ by Ω. Let $t_1, t_2 \in (-\infty, \infty)$ and let $t_2 > t_1$. The symbol $Q_{t_1, t_2} = \Omega \times (t_1, t_2]$ denotes the rectangle. If $t_1 = -\infty$ and $t_2 = +\infty$, then the rectangle Q_{t_1, t_2} becomes a strip which we denote by Q. Every rectangle for which $t_2 - t_1 = \tau_0$, where τ_0 is a fixed number, will be denoted by Q_{τ_0}. From now on we shall assume that $t_1 = 0$ and $t_2 = T$. The closures of Q_{t_1, t_2}, Q and Q_{τ_0} will be denoted by $\overline{Q}_{t_1, t_2}$, $\overline{Q}$ and $\overline{Q}_{\tau_0}$.

The scalar product in the space L_2 of functions in Q_{τ_0} and the norm are given by the expressions

$$(\Phi_1, \Phi_2)_{Q_{\tau_0}} = \int_0^{\tau_0}\int_0^1 \Phi_1\Phi_2\, dx\, dt, \qquad \|\Phi\|_{Q_{\tau_0}}^2 = \int_0^{\tau_0}\int_0^1 \Phi^2\, dx\, dt$$

The scalar product and the norm in L_2 for every $t \in [0, \tau_0]$ will be denoted in similar fashion,

$$(\Phi_1, \Phi_2)_\Omega = \int_0^1 \Phi_1\Phi_2\, dx, \qquad \|\Phi\|_\Omega^2 = \int_0^1 \Phi^2\, dx$$

The Hölder norms for the function $\Phi(x, t)$ defined in Q_{t_1, t_2} are defined as follows:

$$|\Phi|_0 = \sup|\Phi|, \quad |\Phi|_\alpha = |\Phi|_0 + \sup\frac{|\Phi(P_2) - \Phi(P_1)|}{[d(P_1, P_2)]^\alpha}$$
$$|\Phi|_{1+\alpha} = |\Phi|_\alpha + |\Phi_x|_\alpha, \qquad |\Phi|_{2+\alpha} = |\Phi|_{1+\alpha} + |\Phi_t|_\alpha + |\Phi_{xx}|_\alpha$$

$$d(P_1, P_2) = \sqrt{(x'' - x')^2 + |t'' - t'|}, \qquad \alpha \in (0, 1) \quad (P_1, P_2 \in \overline{Q}_{t_1, t_2}) \quad \text{(cont.)}$$

Here P_1 and P_2 are points from Q_{t_1, t_2} with the coordinates (x', t') and (x'', t''), respectively. The function $\Phi(x, t)$ defined in Q_{t_1, t_2} belongs in this domain to the class C^q $(q = 0, \alpha, 1 + \alpha, 2 + \alpha)$ if $|\Phi|_q$ is finite. Finally, we shall use λ_1 to denote the smallest eigenvalue of the boundary value problem

$$v'' + \lambda v = 0, \qquad v(0) = v(1) = 0$$

We shall use the letters M and m accompanied by subscripts to denote constants which depend on the data of the problem and on the domain. In some cases these constants will be given.

1. Periodic solutions of the first boundary value problem. Let us consider the boundary value problem

$$\Phi_t = \nu\,\Phi_{xx} - 2\Phi\Phi_x + \Phi + u\Phi \tag{1.1}$$

$$\frac{du}{dt} + \nu u = -\int_0^1 \Phi^2\, dx \tag{1.2}$$

$$\Phi(0, t) = \psi_1(t), \qquad \Phi(1, t) = \psi_2(t) \tag{1.3}$$

in the domain Q.

In (1.1) the number $\nu > 0$, and the functions ψ_1 and ψ_2 occurring in (1.3) are sufficiently smooth and periodic in t with the period T.

Let (Φ, u), where $\Phi(x, t) \in C^{2+\alpha}$ in $\overline{Q}$ (so that $u(t) \in C^{2+\alpha}$ in $(-\infty, \infty)$), be the solution of boundary value problem (1.1)–(1.3) periodic in t with the period T.

We can obtain several prior estimates for this solution.

Lemma 1.1. The function Φ occurring in the solution of the problem under consideration satisfies the estimate

$$\|\Phi\|_{Q_T}^2 \leqslant M_1 \tag{1.4}$$

Proof. Let (Φ, u) be a periodic solution of problem (1.1)–(1.3). If Φ^2 attains its maximum value on the boundary of Q, then

$$|\Phi_0^2| \leqslant \max(|\psi_1^2|_0, |\psi_2^2|_0) = m_1 \tag{1.5}$$

in $\overline{Q}$, which gives us the estimate

$$\|\Phi\|_{Q_T}^2 \leqslant m_1 T = m_2 \tag{1.6}$$

On the other hand, if Φ^2 has its maximum value at the point (x_0, t_0) inside Q, then the equation obtained by multiplying Eq. (1.1) by Φ yields the inequality

$$-u(t_0) \leqslant 1 \tag{1.7}$$

at the point (x_0, t_0).

But the only periodic solution of Eq. (1.2) is given by the equation

$$u(t) = -\int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}\,\Phi^2(x, \tau)\, dx\, d\tau \tag{1.8}$$

Expressions (1.7), (1.8) imply the inequality

$$\int_{-\infty}^{t_0}\int_0^1 e^{-\nu(t_0-\tau)}\,\Phi^2(x, \tau)\, dx\, d\tau \leqslant 1$$

Replacing the lower limit in the outer integral by $t_0 - T$ and $e^{-\nu(t_0-\tau)}$ by $e^{-\nu T}$, we obtain the estimate

$$\|\Phi\|_{Q_T}^2 \leqslant e^{\nu T} = m_3 \tag{1.9}$$

Expressions (1.6) and (1.9) imply estimate (1.4) with $M_1 = \max(m_2, m_3)$.

Lemma 1.2. The function u occurring in the solution of problem (1.1)–(1.3) satisfies the inequality

$$|u|_0 \leqslant M_2 \tag{1.10}$$

in $[0, T]$ and therefore satisfies it everywhere.

Proof. Instead of Eq. (1.8) we make use of another equation which also defines the periodic solution of Eq. (1.2),

$$u(t) = -\frac{e^{-\nu t-\nu T}}{1-e^{-\nu T}}\int_0^T\int_0^1 e^{\nu\tau}\Phi^2(x,\tau)\,dx\,d\tau - e^{-\nu t}\int_0^t\int_0^1 e^{\nu\tau}\Phi^2(x,\tau)\,dx\,d\tau \tag{1.11}$$

Making use of this equation and estimate (1.4), we obtain the following estimate valid in $[0, T]$:

$$|u|_0 \leqslant \frac{e^{\nu T}}{1-e^{-\nu T}} = M_2$$

Lemma 1.3. For every $t \in (-\infty, \infty)$ we have the estimate

$$\|\Phi\|_\Omega^2 \leqslant M_3 \tag{1.12}$$

Proof. Let us replace the function Φ by the function Ψ by means of the equation

$$\Phi = \Psi + \omega \quad (\omega = \psi_1 + (\psi_2 - \psi_1)\,x) \tag{1.13}$$

The function Ψ is equal to zero at the straight lines $x = 0$ and $x = 1$ and satisfies the equation

$$\Psi_t = \nu\,\Psi_{xx} - 2\Psi\Psi_x - 2\omega\Psi_x - 2\omega_x\Psi - 2\omega\omega_x + \Psi + \omega + u\Psi + u\omega - \omega_t \tag{1.14}$$

in Q.

Let us multiply both sides of this equation by Ψ and integrate the resulting equation over Ω.

$$\frac{d}{dt}\|\Psi\|_\Omega^2 = 2\nu(\Psi, \Psi_{xx})_\Omega - 4(\Psi^2, \Psi_x)_\Omega - 4(\omega\Psi, \Psi_x)_\Omega - 4(\Psi^2, \omega_x)_\Omega - 4(\omega\omega_x, \Psi)_\Omega +$$
$$+ 2\|\Psi\|_\Omega^2 + 2(\omega, \Psi)_\Omega + 2(u, \Psi^2)_\Omega + 2(u\omega, \Psi)_\Omega - 2(\omega_t, \Psi)_\Omega \tag{1.15}$$

Integrating by parts the first three terms in the right side of the above equation and making use of self-evident estimates, we arrive at the inequality

$$dz/dt + \beta\,\|\Psi_x\|_\Omega^2 \leqslant m_4 z + m_5 \quad (\beta = 2\nu,\ z(t) = \|\Psi\|_\Omega^2) \tag{1.16}$$

Since

$$\|\Psi_x\|_\Omega^2 \geqslant \lambda_1\|\Psi\|_\Omega^2,$$

we infer from (1.16) that

$$dz/dt + \beta z \leqslant m_4 z + m_5 \qquad (\beta = 2\nu\lambda_1) \tag{1.17}$$

From (1.17) we find that

$$z(t)\,e^{\beta t} \leqslant z(0) + m_4\int_0^t e^{\beta\tau} z(\tau)\,d\tau + \frac{m_5}{\beta}(e^{\beta t} - 1) \tag{1.18}$$

Setting $t = T$ in this inequality and making use of the periodicity in z, we obtain

$$z(0) \leqslant \frac{m_5}{\beta} + \frac{m_4}{e^{\beta T} - 1}\int_0^T e^{\beta\tau} z(\tau)\,d\tau \tag{1.19}$$

From (1.18), (1.19) we have the inequality

$$z(t) \leqslant \frac{m_5}{\beta} + \frac{m_4 e^{-\beta t}}{-1 + e^{\beta T}}\int_0^T e^{\beta\tau} z(\tau)\,d\tau + m_4 e^{-\beta t}\int_0^t e^{\beta\tau} z(\tau)\,d\tau$$

Making use of estimate (1.4), we infer from this that

$$z(t) \leqslant \frac{m_5}{\beta} + \frac{m_4 M_1 e^{2\beta T}}{e^{\beta T} - 1} \tag{1.20}$$

Applying the relationship between Φ and Ψ and making use of the latter inequality, we obtain estimate (1.12).

Lemma 1.4. The following estimate for the function $u(t)$ is valid in the interval $(-\infty, \infty)$:

$$|du/dt|_0 \leqslant M_4 \tag{1.21}$$

The proof follows directly from the application of estimates (1.10) and (1.12) to Eq. (1.12).

Lemma 1.5. For the function Φ we have the estimate

$$\|\Phi_x\|_{Q_T}^2 \leqslant M_5 \tag{1.22}$$

Proof. Let us multiply both sides of Eq. (1.14) by the function Ψ introduced above by means of Eq. (1.13) and then integrate the resulting equation over the domain Q_T,

$$(\Psi, \Psi_t)_{Q_T} = \nu(\Psi, \Psi_{xx})_{Q_T} - 2(\Psi^2, \Psi_x)_{Q_T} - 2(\omega\Psi, \Psi_x)_{Q_T} - 2(\Psi^2, \omega_x)_{Q_T} - \\ - 2(\omega\omega_x, \Psi)_{Q_T} + \|\Psi\|_{Q_T}^2 + (\omega, \Psi^2)_{Q_T} + (u\omega, \Psi)_{Q_T} - (\Psi, \omega_t)_{Q_T} \tag{1.23}$$

Integrating by parts the first two terms in the right side of the latter equation and availing ourselves of the smoothness of the function ω and the Cauchy inequality in estimating the remaining terms, we arrive at the estimate

$$\|\Psi_x\|_{Q_T}^2 \leqslant m_6 \|\Psi\|_{Q_T}^2 + m_7 \tag{1.24}$$

Applying the relationship between the functions Φ and Ψ to (1.24) and making use of estimate (1.4), we obtain (1.22).

Lemma 1.6. The function Φ satisfies the estimate

$$\int_0^T \|\Phi\|_\Omega^4 \, dt \leqslant M_6 \tag{1.25}$$

The proof of this estimate follows directly from estimate (1.12) and the smoothness property of Φ.

Lemma 1.7. The function Φ satisfies the estimate

$$\|\Phi^2\|_{Q_T}^2 \leqslant M_7 \tag{1.26}$$

Proof.

$$\|\Psi^2\|_{Q_T}^2 \leqslant \int_0^T (\max_x \Psi^2 \|\Psi\|_\Omega^2) \, dt \leqslant \frac{1}{2} \int_0^T (\max_x \Psi^2)^2 \, dt + \frac{1}{2} \int_0^T \|\Psi\|_\Omega^4 \, dt \tag{1.27}$$

Here we have made use of the Cauchy inequality. But

$$\Psi^2 = \int_0^x 2\Psi\Psi_x \, dx \leqslant 2(|\Psi|, |\Psi_x|)_\Omega \tag{1.28}$$

Making use of the Buniakowski inequality, we find that

$$\Psi^2 \leqslant 2\|\Psi\|_\Omega \|\Psi_x\|_\Omega \tag{1.29}$$

Applying (1.29)–(1.27), we obtain the inequality

$$\|\Psi^2\|_{Q_T}^2 \leqslant \int_0^T \|\Psi\|_\Omega^2 \|\Psi_x\|_\Omega^2 \, dt + \frac{1}{2} \int_0^T \|\Psi\|_\Omega^4 \, dt \tag{1.30}$$

This inequality in turn implies the inequality

$$\|\Psi^2\|_{Q_T}{}^2 \leqslant \max_t \|\Psi\|_\Omega{}^2 \|\Psi_x\|_{Q_T}{}^2 + \frac{1}{2}\int_0^T \|\Psi\|_\Omega{}^4\, dt \tag{1.31}$$

Recalling the relationship between the functions Φ and Ψ and making use of estimates (1.12), (1.22) and (1.25), we arrive at estimate (1.26).

Lemma 1.8. The function Φ satisfies the estimate

$$\|\Phi^2\|_\Omega^2 \leqslant M_8 \tag{1.32}$$

Proof. Multiplying Eq. (1.14) by Ψ^3 and integrating the resulting equation over Ω, we obtain

$$\frac{1}{4}\frac{d}{dt}\|\Psi^2\|_\Omega{}^2 = \nu(\Psi^3, \Psi_{xx})_\Omega - 2(\Psi^4, \Psi_x)_\Omega - 2(\omega\Psi^3, \Psi_x)_\Omega - 2(\Psi^4, \omega_x)_\Omega - $$
$$- 2(\omega\omega_x, \Psi^3)_\Omega + \|\Psi^2\|_\Omega{}^2 + (\omega, \Psi^3)_\Omega + (u\Psi, \Psi^3)_\Omega + (u\omega, \Psi^3)_\Omega - (\omega_t, \Psi^3)_\Omega \tag{1.33}$$

Let us consider the first and third terms in the right side of Eq. (1.33),

$$\nu(\Psi^3, \Psi_{xx})_\Omega = -3\nu(\Psi^2, \Psi_x{}^2)_\Omega = -{}^3/_4\nu\|(\Psi^2)_x\|_\Omega{}^2$$

Since $\|(\Psi^2)_x\|_\Omega{}^2 \geqslant \lambda_1\|\Psi^2\|_\Omega{}^2$, it follows that

$$\nu(\Psi^3, \Psi_{xx})_\Omega \leqslant -{}^3/_4\nu\lambda_1\|\Psi^2\|_\Omega{}^2 \tag{1.34}$$

Further,

$$-2(\omega\Psi^3, \Psi_x)_\Omega = -{}^1/_2(\omega, (\Psi^4)_x)_\Omega = {}^1/_2(\omega_x, \Psi^4)_\Omega$$

Hence,

$$2|(\omega\Psi^3, \Psi_x)_\Omega| \leqslant m_8\|\Psi^2\|_\Omega{}^2 \tag{1.35}$$

Estimating the remaining terms in the right side of (1.33) and setting

$$w(t) = \|\Psi^2\|_\Omega^2$$

we obtain the inequality

$$dw/dt + \beta_1 w \leqslant m_9 w + m_{10} \qquad (\beta_1 > 0) \tag{1.36}$$

From (1.36) we find that

$$w(t)e^{\beta_1 t} \leqslant w(0) + m_9\int_0^t e^{\beta_1\tau} w(\tau)\, d\tau + \frac{m_{10}}{\beta_1}(e^{\beta_1 t} - 1) \tag{1.37}$$

Setting $t = T$ in (1.37) and taking into account the periodicity of $w(t)$, we obtain the following estimate for $w(0)$:

$$w(0) \leqslant \frac{m_{10}}{\beta_1} + \frac{m_9 e^{-\beta_1 t}}{e^{\beta_1 T} - 1}\int_0^T e^{\beta_1\tau} w(\tau)\, d\tau \tag{1.38}$$

Replacing the $w(0)$ in (1.37) by the right side of (1.38), we obtain the inequality

$$w(t) \leqslant \frac{m_{10}}{\beta_1} + \frac{m_9 e^{-\beta_1 t}}{e^{\beta_1 T} - 1}\int_0^T e^{\beta_1\tau} w(\tau)\, d\tau + m_9 e^{-\beta_1 t}\int_0^t e^{\beta_1\tau} w(\tau)\, d\tau \tag{1.39}$$

Making use of Eq. (1.13) relating the function Φ and Ψ and applying estimate (1.26), we obtain the required estimate (1.32) from (1.39).

Lemma 1.9. The function Φ satisfies the estimate

$$|\Phi|_0 \leqslant M_9 \tag{1.40}$$

in the domain $\bar{Q}_T$.

Proof. Let $\eta(t)$ be a function with the following properties: $\eta(t)$ is defined for $t \geqslant 0$, has a continuous first-order derivative. $\eta(0) = 0$, and $\eta(t) = 1$ for $t \geqslant \varepsilon$, where $\varepsilon > 0$ is sufficiently small.

The function $\Psi_1 = \eta\Psi$ (see the definition of Ψ above) is the solution of the following boundary value problem in the domain $Q_{0,T+\varepsilon}$:

$$\frac{\partial\Psi_1}{\partial t}=\nu\frac{\partial^2\Psi_1}{\partial x^2}+b(x,t)\frac{\partial\Psi_1}{\partial x}+c(x,t)\Psi_1+f(x,t) \tag{1.41}$$

$$\Psi_1(0,t)=\Psi_1(1,t)=0,\qquad \Psi_1(x,0)=0 \tag{1.42}$$

In Eq. (1.41)

$$b(x,t)=-2\Psi-2\omega,\ c(x,t)=-2\omega_x+1+u$$
$$f(x,t)=\eta(-2\omega\omega_x+\omega+u\omega-\omega_t)+\eta'\Psi \tag{1.43}$$

The functions b, c and f satisfy the estimates

$$\|b^2\|_\Omega^2\leqslant m_{11},\qquad \|c\|_\Omega^2\leqslant m_{12},\qquad \|f\|_\Omega^2\leqslant m_{13} \tag{1.44}$$

which follow directly from estimates (1.10), (1.12) and (1.32). By virtue of Theorem 3, Sect. 3 of [3], the solution of problem (1.41)–(1.43) satisfies the estimate $|\Psi_1|_0\leqslant m_{14}$ in $\overline{Q}_{0,T+\varepsilon}$. This estimate is also valid in the domain $\overline{Q}_{\varepsilon,T+\varepsilon}$, where $\eta=1$. Hence, by virtue of the periodicity of Ψ in $\overline{Q}_T$, we have the estimate $|\Psi|_0\leqslant m_{14}$. This fact and (1.13) imply estimate (1.40).

Lemma 1.10. The function Φ satisfies the following estimate in the domain $\overline{Q}_T$:

$$|\Phi|_\alpha\leqslant M_{10} \tag{1.45}$$

Proof. Estimates (1.44) together with Theorem 5, Sect. 2 of [3] imply the estimate $|\Psi_1|_\alpha\leqslant m_{15}$ (where ψ_1 is the function defined above) in $\overline{Q}_{0,T+\varepsilon}$. Since the function $\Psi_1=\Psi$ for $t\geqslant\varepsilon$, it follows by virtue of the periodicity of Ψ that $|\Psi|_\alpha\leqslant m_{15}$ in $\overline{Q}_T$. This estimate together with (1.13) yields (1.45).

Lemma 1.11. In the domain $\overline{Q}_T$ the function Φ satisfies the estimate

$$|\Phi|_{2+\alpha}\leqslant M_{11} \tag{1.46}$$

Proof. Taking into consideration the estimates (1.21) and (1.45), and applying Theorem 4, Sect. 2 of [4] to the function Φ (as the periodic in t solution of the boundary value problem (1.1), (1.3)), we obtain the estimate (1.46).

We can now formulate and prove the principal theorem of the present section.

Theorem 1.1. If the functions $\psi_1,\psi_2\in C^{2+\alpha}$ in $(-\infty,\infty)$ and if they are periodic in t with the period T, then problem (1.1)–(1.3) has at least one solution (Φ,u) periodic in t with the period T; here $\Phi\in C^{2+\alpha}$ in $\overline{Q}$ and $u\in C^{2+\alpha}$ in $(-\infty,\infty)$.

Proof. We shall prove this theorem with the aid of the Leray-Schauder topological theorem [2] on the existence of fixed points of operator equations.

We denote the space of all functions $\Phi\in C^{1+\alpha}$ in $\overline{Q}$ periodic in t with the period T by $C^{1+\alpha}$ having denoted the norm by the equation $\|\Phi\|=|\Phi|_{1+\alpha}$. We define the operator $A(\Phi,k)$ in the space $C^{1+\alpha}$ for every $k\in[0,1]$ as the operator which associates each function $\Phi\in C^{1+\alpha}$ with a function $\varphi\in C^{2+\alpha}$ which is the solution periodic in t with the period T, of the following boundary value problem:

$$\frac{\partial\varphi}{\partial t}=\nu\frac{\partial^2\varphi}{\partial x^2}+k\left(-2\Phi\frac{\partial\Phi}{\partial x}+\Phi-\Phi\int_{-\infty}^{t}\int_0^1 e^{-\gamma(t-\tau)}\Phi^2(x,\tau)\,dx\,d\tau\right) \tag{1.47}$$

$$\varphi(0,t)=\psi_1(t),\qquad \varphi(1,t)=\psi_2(t) \tag{1.48}$$

By virtue of [4] the function φ exists and belongs to $C^{2+\alpha}$ in $\overline{Q}$. Let us prove the uniform continuity in k of the operator $A(\Phi,k)$ on every bounded set in the space $C^{1+\alpha}$ Let $\Phi_1,\Phi_2,\in C^{1+\alpha}$ and $\|\Phi_1\|,\|\Phi_2\|\leqslant m_{16}$, where m_{16} is some number. If φ_1 and φ_2 are the corresponding periodic solutions of boundary value problem (1.47), (1.48), then the function $\psi=\varphi_1-\varphi_2$ is the periodic solution of the boundary value problem

$$\frac{\partial v^k}{\partial t} = \nu \frac{\partial^2 v^k}{\partial x^2} + k\left[-2\Phi_1\left(\frac{\partial \Phi_1}{\partial x} - \frac{\partial \Phi_2}{\partial x}\right) - 2(\Phi_1 - \Phi_2)\frac{\partial \Phi_2}{\partial x} + \Phi_1 - \Phi_2 - \right. \tag{1.49}$$

$$\left. -(\Phi_1 - \Phi_2)\int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}\Phi_1^2(x, \tau)\,dx\,d\tau - \Phi_2 \int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}(\Phi_1^2(x,\tau) - \Phi_2^2(x,\tau))\,dx\,\tau d\right]$$

$$v^k(0, t) = v^k(1, t) = 0 \tag{1.50}$$

By virtue of [4], the solution v^k satisfies the following estimate in $\bar{Q}$:

$$\| v^k \| = \| \varphi_1 - \varphi_2 \| \leqslant m_{17}\left(| \Phi_1 - \Phi_2 |_0 + \left| \frac{\partial \Phi_1}{\partial x} - \frac{\partial \Phi_2}{\partial x}\right|_0\right) \leqslant m_{17} \| \Phi_1 - \Phi_2 \| \tag{1.51}$$

Here m_{17} depends on m_{16}. Estimate (1.51) implies the uniform continuity in k of the operator $A(\Phi, k)$ on every bounded set in $C^{1+\alpha}$.

Let us prove the complete continuity of the operator $A(\Phi, k)$. Let $\{\Phi_r\}$ be the sequence of functions from $C^{1+\alpha}$ and $\| \Phi_r \| \leqslant m_{18}$, where m_{18} is some number.

We can isolate from this sequence a subsequence $\{\Phi_{r_i}\}$ which converges uniformly together with $\{\partial \Phi_{r_i}/\partial x\}$ in $\bar{Q}$.

Let $\varphi_{r_{i+l}}$ and φ_{r_i} be the solutions of problem (1.49), (1.50) which correspond to the elements $\Phi_{r_{i+l}}$ and Φ_{r_i}. The function

$$V_{r_{i+l}, r_i} = \varphi_{r_{i+l}} - \varphi_{r_i}$$

is clearly a periodic solution of the boundary value problem

$$\frac{\partial V_{r_{i+l}, r_i}}{\partial t} = \nu \frac{\partial^2 V_{r_{i+l}, r_i}}{\partial x^2} + k\left[-2\Phi_{r_{i+l}}\left(\frac{\partial \Phi_{r_{i+l}}}{\partial x} - \frac{\partial \Phi_{r_i}}{\partial x}\right) - 2(\Phi_{r_{i+l}} - \Phi_{r_i})\frac{\partial \Phi_{r_i}}{\partial x} + \right.$$

$$+ \Phi_{r_{i+l}} - \Phi_{r_i} - (\Phi_{r_{i+l}} - \Phi_{r_i})\int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}\Phi_{r_{i+l}}^2(x, \tau)\,dx\,d\tau -$$

$$\left. - \Phi_{r_i}\int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}(\Phi_{r_{i+l}}^2(x,\tau) - \Phi_{r_i}^2(x,\tau))\,dx\,d\tau\right] \tag{1.52}$$

$$V_{r_{i+l}, r_i}(0, t) = V_{r_{i+l}, r_i}(1, t) = 0 \tag{1.53}$$

On the basis of [4] we obtain the following estimate in $\bar{Q}$:

$$\| V_{r_{i+l}, r_i} \| \leqslant m_{19}\left(| \Phi_{r_{i+l}} - \Phi_{r_i} |_0 + \left| \frac{\partial \Phi_{r_{i+l}}}{\partial x} - \frac{\partial \Phi_{r_i}}{\partial x}\right|_0\right)$$

This inequality implies the complete continuity of $A(\Phi, k)$. The operator $A(\Phi, 0)$ maps the entire space $C^{1+\alpha}$ into a single element $\varphi \in C^{2,\alpha}$. This element is the solution in Q of the boundary value problem

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2}, \qquad \varphi(0, t) = \psi_1(t), \qquad \varphi(1, t) = \psi_2(t)$$

periodic in t with the period T .

The transformation $\Phi - A(\Phi, 0)$ is therefore invertible.

If Φ is a fixed point of the operator $A(\Phi, k)$, then $\Phi \in C^{2+\alpha}$ in $\bar{Q}$. The function

$$u(t) = -\int_{-\infty}^{t}\int_0^1 e^{-\nu(t-\tau)}\Phi^2(x, \tau)\,dx\,d\tau$$

is periodic in t with the period T and $u \in C^{2+\alpha}$ in $(-\infty, \infty)$. Thus, the pair of functions (Φ, u) is the solution of the boundary value problem

$$\Phi_t = \nu\Phi_{xx} + k(-2\Phi\Phi_x + \Phi + u\Phi) \quad (1.54)$$

$$\frac{du}{dt} + \nu u = -\int_0^1 \Phi^2(x, t)\,dx, \quad \Phi(0, t) = \psi_1(t), \quad \Phi(1, t) = \psi_2(t) \quad (1.55)$$

periodic in t with the period T.

It is clear that every function $\Phi \in C^{2+\alpha}$ in $\bar{Q}$ periodic in t with period T which occurs in the solution of problem (1.54),(1.55) is also a fixed point of the operator $A(\Phi, k)$. By virtue of Lemma 1.11 and the inequality $\|\Phi\| < \|\Phi\|_{2+\alpha}$, the norms of the possible fixed points of the operator $A(\Phi, k)$ are bounded in aggregate by some number which does not depend on k. All of the conditions of the Leray-Schauder theorem are fulfilled, so that the operator $A(\Phi, k)$ has at least one fixed point for every $k \in [0, 1]$, and specifically for $k=1$. But then the aboye relationship between the fixed points of the operator $A(\Phi, k)$ and the solutions of problem (1.54),(1.55) implies the existence of at least one solution (Φ, u) of problem (1.1)−(1.3) periodic in t with the period T such that $\Phi \in C^{2+\alpha}$ in $\bar{Q}$ and the function u has a continuous first derivative in $(-\infty, \infty)$. Since $\Phi \in C^{2+\alpha}$ in $\bar{Q}$, it is easy to show that $u \in C^{2+\alpha}$ in $(-\infty, \infty)$. Theorem 1.1 has been proved.

2. The first boundary value problem with initial conditions and the Cauchy problem. Let T^* be an arbitrary positive number. Let us consider in Q_{0,T^*} the boundary value problem

$$\Phi_t = \nu\Phi_{xx} - 2\Phi\Phi_x + \Phi + u\Phi, \qquad \frac{du}{dt} + \nu u = -\int_0^1 \Phi^2(x, t)\,dx \quad (2.1)$$

$$\Phi(0, t) = \psi_1(t), \qquad \Phi(1, t) = \psi_2(t), \qquad u(0) = u_0, \ \Phi(x, 0) = \chi(x)$$

Theorem 2.1. If the functions $\psi_1, \psi_2 \in C^{1+\alpha}$ in $[0, T^*]$ and if the function $\chi \in C^{2+\alpha}$ in Ω, then problem (2.1) has a unique solution (Φ, u), where $\Phi \in C^{2+\alpha}$ in $\bar{Q}_{0,T^*}$, and $u \in C^{2+\alpha}$ in the segment $[0, T^*]$.

Theorem 2.2. If $\psi_1 = \psi_2 = 0$ and $\chi \in L_2$ in Ω, then problem (2.1) has a unique generalized solution (Φ, u).

We conclude this section with a consideration of the Cauchy problem

$$\Phi_t = \nu\Phi_{xx} - 2\Phi\Phi_x + \Phi + u\Phi \quad (2.2)$$

$$\frac{du}{dt} + \nu u = -\int_{-\infty}^{+\infty} \Phi^2(x, t)\,dx, \qquad u(0) = u_0, \qquad \Phi(x, 0) = \chi(x)$$

Theorem 2.3. If $\chi \in C^{2+\alpha} \cap L_2$ in $(-\infty, \infty)$, then problem (2.2) has a unique bounded solution (Φ, u), where $\Phi \in C^{2+\alpha}$ in the strip $R = (-\infty, \infty) \times [0, H]$ and $u \in C^{2+\alpha}$ in $[0, H]$.

Theorem 2.4. Let $\chi \in L_2$ in $(-\infty, \infty)$. Cauchy problem (2.2) then has a unique generalized solution (Φ, u).

The proofs of Theorems 2.1 − 2.4 differ only slightly from those of the corresponding theorems of [3, 7, 8, 11], which in turn are based on certain results of [5, 6, 9, 10].

The stability of the solutions of the above turbulence model is investigated in [12].

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RELATION BETWEEN THE LAGRANGIAN AND EULERIAN DESCRIPTIONS OF TURBULENCE

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Let us consider a volume V filled with incompressible fluid. The volume can be either bounded or unbounded. Specifically, the fluid can fill the entire space. The boundaries can vary with time, but this variation must not depend on the motion of the fluid itself. This excludes the stream with a free surface and also the case of a vessel with elastic walls.

The position at the instant t of a fluid particle which initially occupied the position $\mathbf{a}$ will be denoted by $\xi(t, \mathbf{a})$. The condition of incompressibility is

$$\frac{D\xi}{D\mathbf{a}} = 1 \tag{1}$$

The left side of this equation is a transformation Jacobian. The state of the fluid is characterized by the quantities $\sigma^{(k)}(t, \mathbf{a})$, $(k = 1,2,\ldots)$, each of which can denote a set